

# Moufang symmetry III. Integrability of generalized Lie equations

Eugen Paal

## Abstract

Integrability of generalized Lie equations of a local analytic Moufang loop is inquired.  
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## 1 Introduction

In this paper we proceed explaining the Moufang symmetry. The paper can be seen as a continuation of [1, 2].

## 2 Generalized Lie equations

In [1] the *generalized Lie equations* (GLE) of a local analytic Moufang loop  $G$  were found. These read

$$w_j^s(g) \frac{\partial(gh)^i}{\partial g^s} + u_j^s(h) \frac{\partial(gh)^i}{\partial h^s} + u_j^i(gh) = 0 \quad (2.1a)$$

$$v_j^s(g) \frac{\partial(gh)^i}{\partial g^s} + w_j^s(h) \frac{\partial(gh)^i}{\partial h^s} + v_j^i(gh) = 0 \quad (2.1b)$$

$$u_j^s(g) \frac{\partial(gh)^i}{\partial g^s} + v_j^s(h) \frac{\partial(gh)^i}{\partial h^s} + w_j^i(gh) = 0 \quad (2.1c)$$

where  $gh$  is the product of  $g$  and  $h$ , and the auxiliary functions  $u_j^s$ ,  $v_j^s$  and  $w_j^s$  are related with the constraint

$$u_j^s(g) + v_j^s(g) + w_j^s(g) = 0 \quad (2.2)$$

In this paper we inquire integrability of GLE (2.1a–c). Triality [2] considerations are very helpful.

## 3 Generalized Maurer-Cartan equations and Yamagutian

Recall from [1] that for  $x$  in  $T_e(G)$  the infinitesimal translations of  $G$  are defined by

$$L_x \doteq x^j u_j^s(g) \frac{\partial}{\partial g^s}, \quad R_x \doteq x^j v_j^s(g) \frac{\partial}{\partial g^s}, \quad M_x \doteq x^j w_j^s(g) \frac{\partial}{\partial g^s} \quad \in T_g(G)$$

with constraint

$$L_x + R_x + M_x = 0$$

Following triality [2] define the Yamagutian  $Y(x; y)$  by

$$6Y(x; y) = [L_x, L_y] + [R_x, R_y] + [M_x, M_y]$$

We know from [2] the generalized Maurer-Cartan equations:

$$[L_x, L_y] = L_{[x, y]} - 2[L_x, R_y] \quad (3.1a)$$

$$[R_x, R_y] = R_{[y, x]} - 2[R_x, L_y] \quad (3.1b)$$

$$[L_x, R_y] = [R_x, L_y], \quad \forall x, y \in T_e(G) \quad (3.1c)$$

The latter can be written [2] as follows:

$$[L_x, L_y] = 2Y(x; y) + \frac{1}{3}L_{[x, y]} + \frac{2}{3}R_{[x, y]} \quad (3.2a)$$

$$[L_x, R_y] = -Y(x; y) + \frac{1}{3}L_{[x, y]} - \frac{1}{3}R_{[x, y]} \quad (3.2b)$$

$$[R_x, R_y] = 2Y(x; y) - \frac{2}{3}L_{[x, y]} - \frac{1}{3}R_{[x, y]} \quad (3.2c)$$

Define the (secondary) auxiliary functions of  $G$  by

$$\begin{aligned} u_{jk}^s(g) &\doteq u_k^p(g) \frac{\partial u_j^s(g)}{\partial g^p} - u_j^p(g) \frac{\partial u_k^s(g)}{\partial g^p} \\ v_{jk}^s(g) &\doteq v_k^p(g) \frac{\partial v_j^s(g)}{\partial g^p} - v_j^p(g) \frac{\partial v_k^s(g)}{\partial g^p} \\ w_{jk}^s(g) &\doteq w_k^p(g) \frac{\partial w_j^s(g)}{\partial g^p} - w_j^p(g) \frac{\partial w_k^s(g)}{\partial g^p} \end{aligned}$$

The Yamaguti functions  $Y_{jk}^i$  are defined by

$$6Y_{jk}^s(g) \doteq u_{jk}^s(g) + v_{jk}^s(g) + w_{jk}^s(g)$$

Evidently,

$$\begin{aligned} [L_x, L_y] &= -x^j y^k u_{jk}^s(g) \frac{\partial}{\partial g^s} \\ [R_x, R_y] &= -x^j y^k v_{jk}^s(g) \frac{\partial}{\partial g^s} \\ [M_x, M_y] &= -x^j y^k w_{jk}^s(g) \frac{\partial}{\partial g^s} \end{aligned}$$

By adding the above formulae, we get

$$Y(x; y) = -x^j y^k Y_{jk}^s(g) \frac{\partial}{\partial g^s}$$

**Lemma 3.1.** *One has*

$$u_{jk}^i \doteq 2Y_{jk}^i + \frac{1}{3}C_{jk}^s(u^i + 2v^i) \quad (3.3a)$$

$$v_{jk}^i \doteq 2Y_{jk}^i - \frac{1}{3}C_{jk}^s(2u^i + v^i) \quad (3.3b)$$

$$w_{jk}^i \doteq 2Y_{jk}^i + \frac{1}{3}C_{jk}^s(u^i - v^i) \quad (3.3c)$$

*Proof.* To see (3.3a,b) use (3.2a,c) . To see (3.3c) calculate by using (3.2):

$$\begin{aligned}
[M_x, M_y] &= [L_x + R_x, L_y + R_y] \\
&= [L_x, L_y] + [L_x, R_y] + [R_x, L_y] + [R_x, R_y] \\
&= [L_x, L_y] + 2[L_x, R_y] + [R_x, R_y] \\
&= 2Y(x; y) + \frac{1}{3} (L_{[x,y]} - R_{[x,y]})
\end{aligned}$$

and (3.3b) easily follows.  $\square$

## 4 Integrability conditions

**Theorem 4.1.** *The integrability conditons of the GLE (2.1a–c) read*

$$Y_{jk}^s(g) \frac{\partial(gh)^i}{\partial g^s} + Y_{jk}^s(h) \frac{\partial(gh)^i}{\partial g^s} = Y_{jk}^i(gh) \quad (4.1)$$

*Proof.* We differentiate the GLE and use

$$\frac{\partial^2(gh)^i}{\partial g^j \partial g^k} = \frac{\partial^2(gh)^i}{\partial g^k \partial g^j}, \quad \frac{\partial^2(gh)^i}{\partial g^j \partial h^k} = \frac{\partial^2(gh)^i}{\partial g^k \partial h^j}, \quad \frac{\partial^2(gh)^i}{\partial h^j \partial h^k} = \frac{\partial^2(gh)^i}{\partial h^k \partial h^j} \quad (4.2)$$

First differentiate (2.1a) with respect to  $g^p$  and  $h^p$ :

$$\frac{\partial w_j^s(g)}{\partial g^p} \frac{\partial(gh)^i}{\partial g^s} + w_j^s(g) \frac{\partial^2(gh)^i}{\partial g^p \partial g^s} + u_j^s(h) \frac{\partial^2(gh)^i}{\partial g^p \partial h^s} = -\frac{\partial u_j^i(gh)}{\partial(gh)^s} \frac{\partial(gh)^s}{\partial g^p} \quad (4.3a)$$

$$w_j^s(g) \frac{\partial^2(gh)^i}{\partial h^p \partial g^s} + \frac{\partial u_j^s(g)}{\partial h^p} \frac{\partial(gh)^i}{\partial h^s} + u_j^s(h) \frac{\partial^2(gh)^i}{\partial h^p \partial h^s} = -\frac{\partial u_j^i(gh)}{\partial(gh)^s} \frac{\partial(gh)^s}{\partial h^p} \quad (4.3b)$$

Now multiply (4.3a) by  $w_k^p(g)$  and (4.3b) by  $u_k^p(g)$  and add the resulting formulae. On the right hand side of the resulting formula use again the GLE (2.1a); then transpose the indexes  $j$  and  $k$  and subtract the result from the previous one. Then it turns out that due to (4.2) all terms with the second order partial derivatives vanish and result reads

$$w_{jk}^s(g) \frac{\partial(gh)^i}{\partial g^s} + u_{jk}^s(h) \frac{\partial(gh)^i}{\partial h^s} = u_{jk}^i(gh) \quad (4.4)$$

By acting analogously with GLE (2.1b,c) we get

$$v_{jk}^s(g) \frac{\partial(gh)^i}{\partial g^s} + w_{jk}^s(h) \frac{\partial(gh)^i}{\partial h^s} = v_{jk}^i(gh) \quad (4.5a)$$

$$u_{jk}^s(g) \frac{\partial(gh)^i}{\partial g^s} + v_{jk}^s(h) \frac{\partial(gh)^i}{\partial h^s} = w_{jk}^i(gh) \quad (4.5b)$$

Now add (4.4), (4.5a) and (4.5b) to obtain (4.1).

It remains to show that (4.4), (4.5a) and (4.5b) are equivalent to (4.1). By using (3.3a–c) calculate

$$\begin{aligned}
& w_{jk}^s(g) \frac{\partial(gh)^i}{\partial g^s} + u_{jk}^s(h) \frac{\partial(gh)^i}{\partial h^s} - u_{jk}^i(gh) \stackrel{(3.3a,c)}{=} \\
& v_{jk}^s(g) \frac{\partial(gh)^i}{\partial g^s} + w_{jk}^s(h) \frac{\partial(gh)^i}{\partial h^s} - v_{jk}^i(gh) \stackrel{(3.3b,c)}{=} \\
& u_{jk}^s(g) \frac{\partial(gh)^i}{\partial g^s} + v_{jk}^s(h) \frac{\partial(gh)^i}{\partial h^s} - w_{jk}^i(gh) \stackrel{(3.3a-c)}{=} \\
& = 2 \left( Y_{jk}^s(g) \frac{\partial(gh)^i}{\partial g^s} + Y_{jk}^s(h) \frac{\partial(gh)^i}{\partial g^s} - Y_{jk}^i(gh) \right) \quad \square
\end{aligned}$$

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## References

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Department of Mathematics  
Tallinn University of Technology  
Ehitajate tee 5, 19086 Tallinn, Estonia  
E-mail: eugen.paal@ttu.ee